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Translated by A. Y.

PERIODIC SOLUTIONS OF SECOND ORDER DYNAMIC SYSTEMS CLOSE TO PIECE-WISE HAMILTONIAN SYSTEMS

PMM Vol. 33, №5, 1969, pp. 912-915

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(Received May 6, 1969)

We show the conditions which must be satisfied by the approximating functions, in order that the result known for the nearly Hamiltonian systems with the analytic right sides [1] would also hold for the systems with piece-wise analytic right sides.

Theorem. Let $H(x, y) = h$ be a family of closed curves C_h dependent on the parameter h, and matched from segments $H_i(x, y) = h$ on the intervals $x_i \leqslant x \leqslant x_{i+1}$. Functions $H_i(x, y)$ are analytic in each of their arguments.

Then a unique limit cycle exists in the neighborhood of the closed curve C_{h_0}, for the

system $$x^\cdot = H_y'(x, y) + \mu p(x, y), \quad y^\cdot = -H_x'(x, y) + \mu q(x, y) \tag{1}$$
when $\mu \neq 0$, provided that $\partial H / \partial y$ is continuous at the points of matching $x = x_i$.

Here $p(x, y)$ and $q(x, y)$ are functions, analytic on each of the intervals $x_i \leqslant x \leqslant x_{i+1}$, and h_0° is a root of the equation

$$\Psi(h^\circ_0) \equiv \int_{C_{h_0^\circ}} q(x, y)\,dx - p(x, y)\,dy = 0, \quad \Psi'(h_0^\circ) \neq 0$$

The limit cycle will be stable when $\Psi'(h_0^\circ) < 0$ and unstable when $\Psi'(h_0^\circ) > 0$.

Proof. Let denote by $S_i^{(1)}$ the half-lines $x = x_i$ for $y > 0$, and by $S_i^{(2)}$ the half-lines $x = x_i$ for $y < 0$ and let us consider the phase trajectories of the system (1) for the cases $\mu = 0$ and $\mu \neq 0$, both satisfying the same initial conditions

$$x = x_0, \quad y = y_0 \text{ when } t = 0 \tag{2}$$

Assuming that the trajectory of the system (1) satisfying the conditions (2) intersects the half-lines $S_k^{(j)}$ at the points $P_{k0}^{(j)}(x_k, y_{k0}^{(j)})$ when $\mu = 0$ and at the points $P_k^{(j)}(x_k, y_k^{(j)})$ when $\mu \neq 0$, we shall first prove that a point transformation of the half-line $S_0^{(1)}$ into the half-line $S_k^{(1)}$ has the form

$$y_k^{(1)} = y_{k0}^{(1)} + \frac{\mu}{\partial H_{k-1}(x_k, y_{k0}^{(1)})/dy} \int_{(x_0, y_0)}^{(x_k, y_{k0}^{(1)})} q(x, y)\,dx - p(x, y)\,dy + \mu^2(\ldots) \tag{3}$$

provided that the function $\partial H / \partial y$ is continuous at $x = x_i$.

Let us consider the point transformation of the half-line $S_0^{(1)}$ into the half-line $S_1^{(1)}$. When $\mu = 0$, solution of (1) satisfying the conditions (2) can be written as

$$x = x_0(h_0, \ t + \varphi_0), \quad y = y_0(h_0, \ t + \varphi_0) \quad (x_0 < x < x_1) \tag{4}$$
$$(h_0, \ \varphi_0 = \text{const})$$

We shall seek a solution of the system (1) in the case of $\mu \neq 0$ in the form

$$\begin{aligned} x &= x_0[\alpha_0(t), \ t + \beta_0(t)] \equiv \xi_0(t) \\ y &= y_0[\alpha_0(t), \ t + \beta_0(t)] \equiv \eta_0(t) \end{aligned} \tag{5}$$

where $\alpha_0(t)$ and $\beta_0(t)$ are some functions of time t.

Inserting (5) into (1) we obtain

$$\begin{aligned} \frac{\partial x_0}{\partial \alpha_0}\frac{d\alpha_0}{dt} + \frac{\partial x_0}{\partial \beta_0}\frac{d\beta_0}{dt} &= \mu p[\xi_0(t), \eta_0(t)] \\ \frac{\partial y_0}{\partial \alpha_0}\frac{d\alpha_0}{dt} + \frac{\partial y_0}{\partial \beta_0}\frac{d\beta_0}{dt} &= \mu q[\xi_0(t), \eta_0(t)] \end{aligned} \tag{6}$$

Taking into account that

$$\frac{\partial x_0}{\partial \beta_0} = \frac{\partial H_0[\xi_0(t), \eta_0(t)]}{\partial y}, \quad \frac{\partial y_0}{\partial \beta_0} = -\frac{\partial H_0[\xi_0(t), \eta_0(t)]}{\partial x}$$
$$\left(\frac{\partial H_0}{\partial x}\frac{\partial x_0}{\partial h_0} + \frac{\partial H_0}{\partial y}\frac{\partial y_0}{\partial h_0}\right)_{x=\xi_0(t),\, y=\eta_0(t)} \equiv 1$$

we obtain from (6),

$$\begin{aligned} \frac{d\alpha_0}{dt} &= \mu\left\{q[\xi_0(t), \eta_0(t)]\frac{\partial x_0}{\partial t} - p[\xi_0(t), \eta_0(t)]\frac{\partial y_0}{\partial t}\right\} \\ \frac{d\beta_0}{dt} &= \mu\left\{p[\xi_0(t), \eta_0(t)]\frac{\partial y_0}{\partial h_0} - q[\xi_0(t), \eta_0(t)]\frac{\partial x_0}{\partial h_0}\right\} \end{aligned} \tag{7}$$

Functions $\alpha_0(t)$ and $\beta_0(t)$ should satisfy the following initial conditions:

$$\alpha_0(t) = h_0, \quad \beta_0(t) = \varphi_0 \text{ when } t = 0$$

Writing $\alpha_0(t)$ and $\beta_0(t)$ in the form of power series in μ, we obtain

$$\alpha_0(t) = h_0 + \mu\alpha_{01}(t) + \mu^2(\ldots), \qquad \beta_0(t) = \varphi_0 + \mu\beta_{01}(t) + \mu^2(\ldots) \tag{5}$$

$$\alpha_{01}(t) = \int_0^t \left\{ q[\xi_0(t), \eta_0(t)] \frac{\partial x_0}{\partial t} - p[\xi_0(t), \eta_0(t)] \frac{\partial y_0}{dt} \right\}_{\mu=0} dt$$

(In the following, the expression for β_{01} (t) shall not be required).

Let $t = t_1$ be the shortest time in which the representative point moving along the trajectory of (1) can reach the half-line $S_1^{(1)}$ at the point $(x_1, y_1^{(1)})$.

Substituting $t = t_1$ into (5) and expanding the resulting relations into power series in μ, we obtain

$$t_1 = t_{10} + \mu t_{11} + \mu^2(\ldots)$$

$$x_1 = x_0(h_0, t_{10} + \varphi_0) + \mu \left\{ \frac{\partial x_0}{\partial t}[t_{11} + \beta_{01}(t)] + \frac{\partial x_0}{\partial h_0}\alpha_{01}(t) \right\}_{t=t_1,\ \mu=0} + \mu^2(\ldots)$$

$$y_1^{(1)} = y_{10}^{(1)} + \mu \left\{ \frac{\partial y_0}{\partial t}[t_{11} + \beta_{01}(t)] + \frac{\partial y_0}{\partial h_0}\alpha_{01}(t) \right\}_{t=t_1,\ \mu=0} + \mu^2(\ldots)$$

which, on eliminating $t_{11} + \beta_{01}(t_{10})$, yield

$$y_1^{(1)} = y_{10}^{(1)} + \frac{\mu}{\partial H_0(x_1, y_{10}^{(1)}) / \partial y} \int_{(x_0, y_0)}^{(x_1, y_{10}^{(1)})} q(x, y)\, dx - p(x, y)\, dy + \ldots \tag{9}$$

The integral is taken along the curve of the system (1) passing through the point (x_0, y_0) with $\mu = 0$.

Let us now consider the point transformation, taking the half-line $S_0^{(1)}$ into the half-line $S_2^{(1)}$. We shall represent the solution of (1) satisfying the conditions

$$x = x_1, \quad y = y_1^{(1)} \text{ when } t = t_1 \tag{10}$$

in the form

$$x = x_1(h_1, t + \varphi_1), \quad y = y_1(h_1, t + \varphi_1) \tag{11}$$

when $\mu = 0$, and in the form

$$x = x_1[\alpha_1(t), t + \beta_1(t)] \equiv \xi_1(t), \quad y = y_1[\alpha_1(t), t + \beta_1(t)] \equiv \eta_1(t) \tag{12}$$

when $\mu \neq 0$.

Writing $\alpha_1(t)$ and $\beta_1(t)$ in the form of power series in μ, we obtain

$$\alpha_1(t) = h_1 + \mu\alpha_{11}(t) + \mu^2(\ldots), \quad \beta_1(t) = \varphi_1 + \mu\beta_{11}(t) + \mu^2(\ldots) \tag{13}$$

$$\alpha_{11}(t) = \int_{t_1}^{t} \left\{ q[\xi_1(t), \eta_1(t)] \frac{\partial x_1}{\partial t} - p[\xi_1(t), \eta_1(t)] \frac{\partial y_1}{\partial t} \right\}_{\mu=0} dt$$

Let $t = t_2$ be the instant of time, at which the representative point moving along the trajectory of (1) reaches the half-line $S_2^{(1)}$.

Inserting $t = t_2$ into (12) and expanding the resulting expressions into power series in μ, we have

$$t_2 = t_{20} + \mu t_{21} + \mu^2(\ldots)$$

$$h_1 = h_{10} + \mu h_{11} + \mu^2(\ldots), \quad \varphi_1 = \varphi_{10} + \mu\varphi_{11} + \mu^2(\ldots)$$

$$y_2^{(1)} = y_{20}^{(1)} + \frac{\mu}{\partial H_1(x_2, y_{20}^{(1)}) / \partial y} [h_{11} + \alpha_{11}(t_{20})] + \mu^2(\ldots)$$

from which, taking into account the fact that

$$h_1 = H_1(x_1, y^{(1)}), \quad \frac{\partial h_1}{\partial \mu} = h_{11} = \frac{\partial H_1(x_1, y_{10}^{(1)})}{\partial y} \left. \frac{\partial y_1^{(1)}}{\partial \mu} \right|_{\mu=0}$$

and using (9), we obtain

$$y_2^{(1)} = y_{20}^{(1)} + \frac{\mu}{\partial H_1(x_2, y_{20}^{(1)}) / \partial y} \Bigg[\int\limits_{(x_1, y_{10}^{(1)})}^{(x_2, y_{20}^{(1)})} q(x,y)\,dx - p(x,y)\,dy +$$

$$+ \frac{\partial H_1(x_1, y_{10}^{(1)}) / \partial y}{\partial H_0(x_1, y_{10}^{(1)}) / \partial y} \int\limits_{(x_0, y_0)}^{(x_1, y_{10}^{(1)})} q(x,y)\,dx - p(x,y)\,dy \Bigg] + \mu^2(\ldots) \qquad (14)$$

Here the integrals are taken along the curve C_{h_0} passing through the point $P_0(x_0, y_0)$, and $h_0 = H_0(x_0, y_0)$.

If the function $\partial H / \partial y$ is continuous at $x = x_i$, then

$$\partial H_1(x_1, \ y_{10}^{(1)}) / \partial y = \partial H_0(x_1, \ y_{10}^{(1)}) / \partial y$$

and the expression (14) can be written in the form of (3) with $k = 2$.

Assuming now that the formula (3) is true for the transformation of the half-line $S_0^{(1)}$ into the half-line $S_{k-1}^{(1)}$, we can show that it is also true for the transformation of $S_0^{(1)}$ into $S_k^{(1)}$, provided that the function $\partial H / \partial y$ is continuous at $x = x_i$.

Similarly, assuming the continuity of the function $\partial H / \partial y$ we can show that relation (3) holds for the transformation of $S_0^{(1)}$ into $S_k^{(2)}$ (with the representative point passing through the straight line $y = 0$), provided that $\partial H_{k-1}(x_k, \ y_{k0}^{(1)}) / \partial y$ is replaced by $\partial H_k(x_k, y_{k0}^{(2)}) / \partial y$, and the superscripts $^{(1)}$ by $^{(2)}$.

Everything that has been said above concerning the transformation of the half-line $S_0^{(1)}$ into $S_k^{(2)}$, also holds for transformation of the half-line $S_k^{(2)}$ in the lower semiplane into the initial half-line $S_0^{(1)}$ in the upper semiplane.

Point transformation of the half-line $S_0^{(1)}$ into itself in the neighborhood of the closed curve C_{h_0} passing through the point $P_0(x_0, \ y_0)$, has the form

$$y_0^{(1)} = y_0 + \frac{\mu}{\partial H_0(x_0, y_0) / \partial y} \int\limits_{C_{h_0}} q(x,y)\,dx - p(x,y)\,dy + \qquad (15)$$

$$+ \mu^2(\ldots) \equiv y_0 + \frac{\mu}{\partial H_0(x_0, y_0) / \partial y} \Psi(h_0) + \mu^2(\ldots)$$

Clearly, if

$$\Psi(h_0^\circ) = 0, \ \ \Psi'(h_0^\circ) \neq 0$$

then the transformation (15) has a unique fixed point $P_0(x_0, \ y_0^\circ + \mu y_1)$, which tends to the point $P(x_0, \ y_0^\circ)$ as $\mu \to 0$ $(h_0^\circ = H_0(x_0, \ y_0^\circ))$.

At the same time system (1) has a unique limit cycle situated near the curve $C_{h_0^\circ}$, which tends to this curve for $\mu \to 0$.

Koenigs' theorem [2] implies that the fixed point $P_0(x_0, \ y_0^\circ + \mu y_1)$ and the corresponding limit cycle are stable if $\Psi'(h_0^\circ) < 0$ and unstable, if $\Psi'(h_0^\circ) > 0$.

If the functions $\partial H/\partial x$, $\partial H/\partial y$, $p(x,y)$ and $q(x,y)$ are 2π-periodic in x then the phase space of the system (1) will be periodic with two straight lines $x = x_0$ and $x = x_0 + 2\pi$ coinciding. The theorem proved above gives, in this case, the conditions of existence and stability of the limit cycle of (1) enveloping the phase cylinder.

The author expresses her gratitude to N. N. Bautin for valuable advice.

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Translated by L. K.